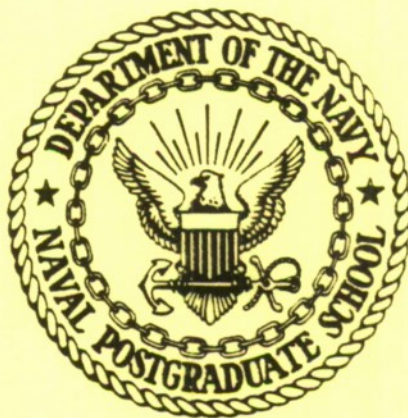


1094  
NPS55-79-010

# NAVAL POSTGRADUATE SCHOOL

## Monterey, California



HIDE AND SEEK FROM A FIXED BASE

by

Alan R. Washburn

April 1979

QA  
270  
W296

Approved for public release; distribution unlimited.

20091105043

QA  
270  
W296

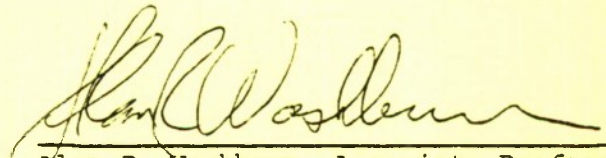
NAVAL POSTGRADUATE SCHOOL  
MONTEREY, CALIFORNIA

Rear Admiral T. F. Dedman  
Superintendent

J. R. Borsting  
Provost

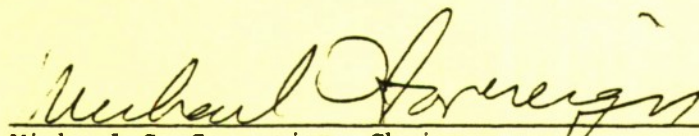
Reproduction of all or part of this report is authorized.

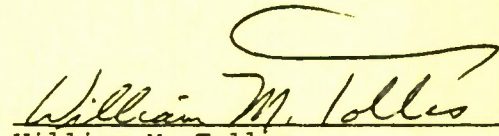
This report was prepared by:

  
Alan R. Washburn, Associate Professor  
Department of Operations Research

Reviewed by:

Released by:

  
Michael G. Sovereign, Chairman  
Department of Operations Research

  
William M. Tolles  
Dean of Research



UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS55-79-010	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle)  Hide and Seek From a Fixed Base		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s)  Alan R. Washburn		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, Ca. 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, CA. 93940		12. REPORT DATE April 1979
		13. NUMBER OF PAGES 18
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEOULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release, distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Game Search Hide		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  An abstract hide and seek game is solved, the unique feature of which is that the hiders are constrained to return to a fixed point periodically.		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE  
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# HIDE AND SEEK FROM A FIXED BASE

by

Alan R. Washburn  
Naval Postgraduate School  
Monterey, CA 93940

## Background

Suppose that a force of "hiders" has an area  $A$  available within which to hide from a force of "seekers." Each side can distribute its forces arbitrarily within  $A$ . If the density of seeker effort at the location of any given hider is  $s$ , then the hider is assumed to escape with probability  $f(s)$ , where  $f(s)$  is some decreasing, convex function of  $s$ . The hiders want to maximize the probability of escape, and the seekers want to minimize it. If no further constraints are imposed, it is not difficult to show that each side should allocate its forces uniformly throughout the region, and that the escape probability is the value of a two-person zero sum game is  $f(S/A)$ , where  $S$  is the total amount of seeker effort. Our object in this report is to investigate the impact of constraints on the motion of the hiders. Specifically, we want to investigate what happens if the hiders are required to visit a particular point (a port, typically) on the boundary of  $A$  every  $t$ , while never travelling at a speed exceeding  $v$ .

We can anticipate that the escape probability will be  $f(S/A)$  when the product  $vt$  is "large," and 0 when  $vt$  is "small."

In order to simplify the analysis, the following assumptions are made:

- a) the region is a circular sector (wedge)
- b) the revisit point is at the apex of the wedge
- c)  $f(s) = 1/(1 + s)$ .

In Figure 1, the heaviness of the shading indicates the density of seeker effort for a typical seeker strategy. Note that the effort is dense in the vicinity of the apex, since the hiders must all transit through that area in order to visit the apex. A typical hider "tour" is shown. The hider basically picks a direction at random and a range from a distribution introduced in the next section, goes to the point and stays in the vicinity of that point until it is time to return to the apex. Each hider picks an independent tour after each visit to the apex.

## Results

Let

$\theta$  = angle of the circular sector

$r_{\text{sec}}$  = radius of the circular sector

$A = \theta r_{\text{sec}}^2 / 2$  = area of sector

$v$  = hider speed

$t$  = revisit time

$r_{\text{max}} = vt/2$  = maximum range of the hiders

$S$  = total amount of seeker effort

$$Y = S/(\theta r_{\max}^2)$$

$$U = r_{\text{sec}}/r_{\max}$$

The value of the game (escape probability) depends on the two dimensionless quantities  $Y$  (a normalized amount of seeker effort) and  $U$  (a normalized sector size); call it  $P(U,Y)$ . Figure 2 shows  $P(U,Y)$  as a function of  $Y$  for several values of  $U$ . Since  $U = 1$  corresponds to the case where the maximum range of the hiders is equal to the sector radius, all values of  $U > 1$  follow the same curve as for  $U = 1$ . The curves in Figure 2 are equivalent to:

$$(1) \quad \text{Let } V = 1 - \sqrt{1 - U^2}$$

$$\text{Case 1: for } Y \leq V^2/6, P(U,Y) = 1 - \sqrt{2Y/3}$$

$$(2) \quad \text{Case 2: for } Y \geq V^2/6, P(U,Y) = V/(Y + U^2/2)$$

Standard limiting operations show that

$$(3) \quad \lim_{r_{\max} \rightarrow 0} P(U,Y) = 0$$
$$\lim_{r_{\max} \rightarrow \infty} P(U,Y) = 1/(1 + S/A) ,$$

as anticipated.

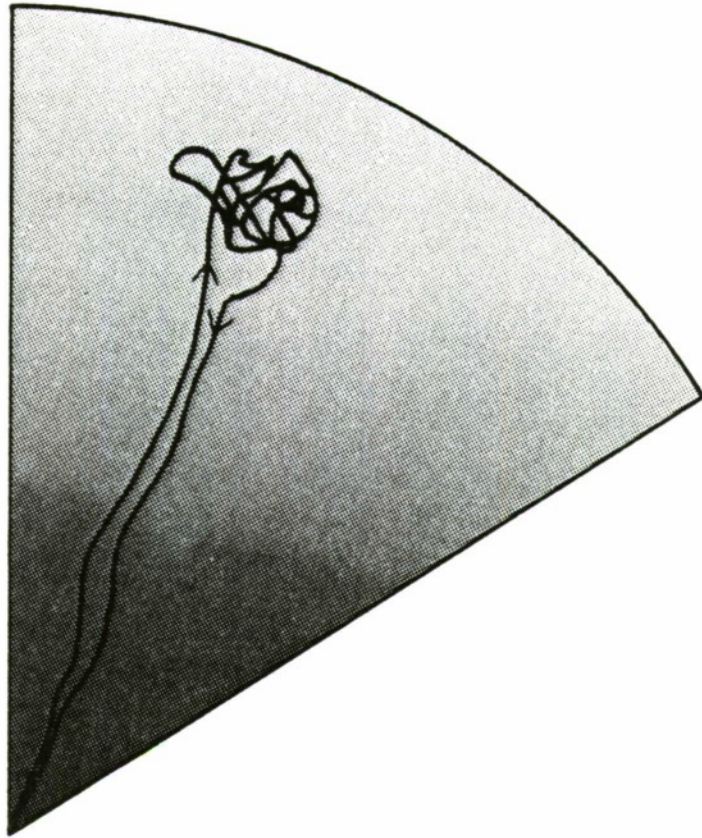


FIGURE 1  
ILLUSTRATING A HIDER PATH AND A SHADED SEEKER DENSITY



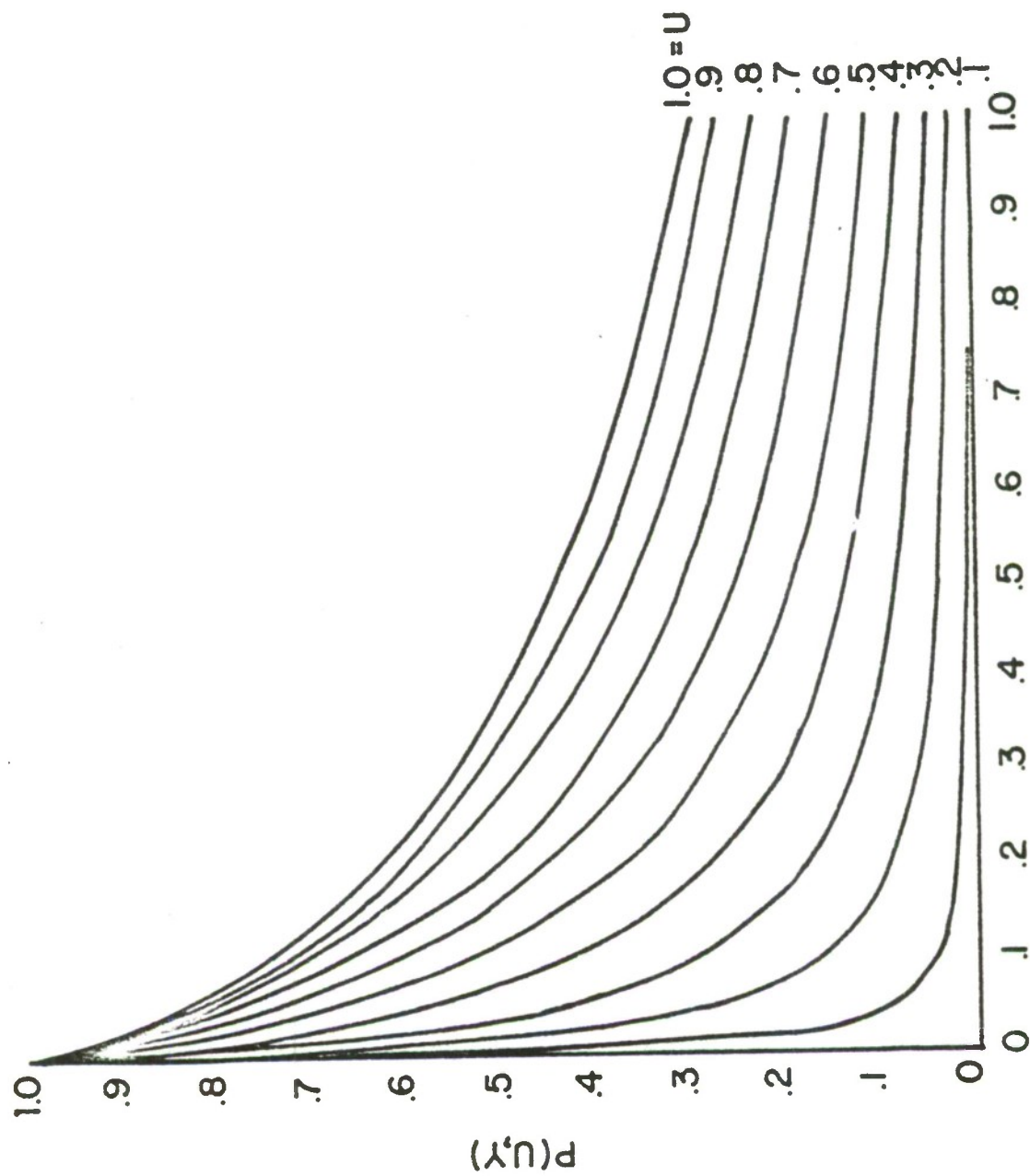


FIGURE 2. PROBABILITY OF SURVIVAL vs  $y$  FOR VARIOUS  $u$ .



For example, suppose

$$r_{\max} = 5000 \text{ mi}$$

$$r_{\text{sec}} = 1000 \text{ mi}$$

$$S = (25 \text{ hr holding time}) (10^5 \text{ sq mi/hr search rate})$$

$$= 25 \times 10^5 \text{ sq mi}$$

$$\theta = 1 \text{ radian}$$

Then  $Y = .1$ ,  $U = .2$ , and  $V = .02$ . This is Case 2, and  $P(U, Y) = .164$ . If  $r_{\max}$  were "very large," we would have  $P(U, Y) = 1/6$ .

The optimal strategy for the hiders is to pick an angle at random and a range from the distribution  $F(ur_{\max})$ , where

$$(4) \quad F(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq V \\ \frac{(u-V)^2}{2V(1-u)} & \text{for } V \leq u \leq U, \end{cases}$$

and where  $V$  is as earlier defined. Qualitatively, the hiders have a tendency to pick large ranges, with  $Vr_{\max}$  being the smallest range picked.

Let  $y(u)$  be the density of seeker effort at range  $ur_{\max}$ , let  $T = \sqrt{6Y}$ , and let  $K = (Y + U^2/2)/[V(1 - V/3)]$ . Then the optimal density  $y(u)$  is

$$(5) \text{ Case 1. } y(u) = \begin{cases} \sqrt{T/u} - 1 & \text{for } u \leq T \\ 0 & \text{for } u \geq T \end{cases}$$

$$(6) \text{ Case 2. } y(u) = \begin{cases} K \sqrt{V/u} - 1 & \text{for } u \leq V \\ K - 1 & \text{for } u \geq V \end{cases}$$

Qualitatively, the searchers have a tendency to cluster near the apex, particularly in Case 1, ( $V \leq V^2/6$ ). The density is actually unbounded near the apex; that is,  $\lim_{u \rightarrow 0} y(u) = \infty$ .

The proof that the functions given above represent the value of the game and the optimal strategies for the two sides is the subject of the next section.

#### Exact Statement of the Problem

Let  $u$  be range from the apex measured in units of  $r_{\max}$ , so that the hiders must pick a range  $u$  for each tour in the interval  $[0, U]$ . Let  $F(v)$  be the C.D.F. used by the hiders. Then the hiders spend  $G(u)$  of their time within  $u$  of the apex, where

$$(7) \quad G(u) = F(u) + u(1 - F(u)) .$$

Formula (7) is true because a hider will be within  $u$  of the apex throughout its patrol period if it picks a range smaller than or equal to  $u$ , and will spend a fraction  $u$  of its patrol time within  $u$  of the apex even if it picks a range greater than  $u$ .  $F(u)$  can be any C.D.F. defined on  $[0, U]$ , but  $G(u)$  cannot, which is what makes the problem non-trivial.

Since  $y(u)$  is the density of seeker effort at range  $u$ , the escape probability for a hider averaged over time is

$$(8) \quad A(F, y) = \int_0^U (1 + y(u))^{-1} dG(u) ,$$

where

$$(9) \quad \int_0^U y(u) u du = Y \quad \text{and} \quad y(u) \geq 0 .$$

Equations (7), (8), and (9) define a two-person zero sum game where the hiders select a C.D.F.  $F(u)$  on  $[0, U]$  and the seekers select  $y(u)$  according to (9). We next show that the results quoted earlier constitute a saddle point of this game.

### Proof of Results

The results shown below were discovered by using the theory of optimal control, but we will prove that the game has been solved by showing that the solution offered is a saddle point. While this is analytically simpler, it will not motivate the results.

We must show that

$$\max_F A(F, y^*) = P(U, Y) = \min_y A(F^*, y) ,$$

where  $F^*$  and  $y^*$  are the functions given earlier.

Proof that  $P(U, Y) = \min_y A(F^*, y)$

Let  $G^*(u) = F^*(u) + u(1 - F^*(u))$ . Using (1) and (4),  $F^*(U) = 1$ , so also  $G^*(U) = 1$ . After substitution and simplification,

$$G^*(u) = \begin{cases} u & \text{for } u \leq V \\ \frac{u^2 + V^2}{2V} & \text{for } V \leq u \leq U \end{cases}$$

Let  $g(u) = (d/du)G^*(u)$ . Then we have

$$A(F^*, y) = \int_0^U g(u)/(1 + y(u)) du ,$$

where

$$g(u) = \begin{cases} 1 & \text{for } u \leq V \\ u/V & \text{for } V \leq u \leq U \end{cases}$$

Consider the Lagrangian



$$h(y) = \int_0^U \left[ \frac{g(u)}{1 + y(u)} + \lambda u y(u) \right] du ,$$

which is to be minimized subject to  $y(u) \geq 0$ . We minimize for each  $u$  separately by differentiation, obtaining the minimizing function  $\tilde{y}$ :

$$\tilde{y}(u) = \left( \sqrt{\frac{g(u)}{\lambda u}} - 1 \right)^+ ,$$

where  $+$  indicates that  $\tilde{y}(u)$  is to be 0 rather than negative. If  $\lambda V < 1$ ,  $y(u) > 0$  for all  $u$ , and

$$\begin{aligned} (10) \quad A(F^*, \tilde{y}) &= \int_0^U \sqrt{\lambda u g(u)} du \\ &= \sqrt{\lambda} \left[ \int_0^V \sqrt{u} du + \int_V^U u/\sqrt{V} du \right] \\ &= \sqrt{\lambda/V} \left[ \frac{2}{3} V^2 + (U^2 - V^2)/2 \right] \\ &= \sqrt{\lambda/V} [V^2/6 + U^2/2] \end{aligned}$$

If  $\tilde{y}$  is to be feasible, we must also have

$$\begin{aligned} (11) \quad Y &= \int_0^U u \tilde{y}(u) du = \int_0^V \sqrt{u/\lambda} du + \int_V^U \frac{u}{V \sqrt{\lambda V}} du - U^2/2 \\ &= \frac{1}{\sqrt{\lambda V}} [V^2/6 + U^2/2] - U^2/2 \end{aligned}$$

Since  $U^2 + V^2 = 2V$ ,  $V^2/6 + U^2/2 = V(1 - V/3)$ . Solving (11) for  $\sqrt{\lambda}$  and then substituting  $\sqrt{\lambda}$  in (10), we obtain  $A(F^*, \tilde{y}) = P(U, Y)$ , and also  $\lambda V < 1$  if and only if  $Y > V^2/6$ .

If  $\lambda V \geq 1$ ,  $\tilde{y}(u) = 0$  for  $u \geq 1/\lambda$ . Let  $T = 1/\lambda$ . Then

$$(12) \quad A(F^*, \tilde{y}) = \int_0^T \sqrt{\lambda u} \, du + 1 - T = 1 - T/3 ,$$

and if  $\tilde{y}$  is to be feasible we must have

$$(13) \quad Y = \int_0^T u(\sqrt{1/\lambda u} - 1) \, du = 2T^2/3 - T^2/2 = T^2/6$$

Solving (13) for  $T$  and substituting in (12), we obtain

$A(F^*, \tilde{y}) = P(U, Y)$ , and also  $\lambda V \geq 1$  if and only if  $Y \leq V^2/6$ .

According to Everett's theorem [1] on Lagrange multipliers,  $A(F^*, y) \geq A(F^*, \tilde{y})$ , so we have shown that  $P(U, Y) = \min_y A(F^*, y)$ . We also note that  $\tilde{y} = y^*$ .

Proof that  $P(U, Y) = \max_F A(F, y^*)$

Since  $y^*(u)$  is differentiable, we can integrate  $A(F, y^*)$  by parts to obtain

$$A(F, y^*) = \frac{G(U)}{1 + y^*(U)} - \int_0^U G(u) B(u) \, du ,$$

where

$$B(u) = - \left( \frac{d}{du} y^*(u) \right) / (1 + y^*(u))^2 .$$

In both Cases I and II,  $B(u) \geq 0$  for  $u \leq V$ , and  $B(u) = 0$  for  $u \geq V$  (note  $T \leq V$  in (5)). Since  $G(u) = u + F(u)(1-u) \geq u$ ,

$$A(F, y^*) \leq \frac{1}{1 + y^*(U)} - \int_0^U uB(u) du .$$

But it is also true that  $G^*(u) = u$  for  $u \leq V$ , so  $A(F, y^*) \leq A(F^*, y^*)$  for any  $F$ . But we already know  $A(F^*, y^*) = P(U, Y)$ , so the proof is complete.

ACKNOWLEDGMENT. This research was conducted while acting as consultant to ORI Inc. and reported on separately.

#### REFERENCE

- (1) Everett, H. III, "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources," *Opns. Res.* 11, 399-417 (1963).

# INITIAL DISTRIBUTION LIST

## COPIES

Dean of Research Code 012 Naval Postgraduate School Monterey, California 93940	1
Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
Commanding Officer Air Test and Evaluation Squadron 1 (VX-1) Patuxent River, Maryland 20670 Attn: Code 713	1
Commanding Officer Submarine Development Group Two Groton, Connecticut 06340	1
Director Strategic Systems Project Office 1931 Jefferson Davis Highway Arlington, Virginia 20376 Attn: Code SP2021	1
Naval Air Development Center Code 2022 Johnsville, Pennsylvania 18974	1
Center for Naval Analysis 1401 Wilson Boulevard Arlington, Virginia 22209	1
Naval Weapons Laboratory Dahlgren, Virginia 22448	1
Naval Weapons Center China Lake, California 93555	1
Naval Surface Weapons Center White Oak Silver Spring, Maryland 20910	1
Naval Research Laboratory Washington, D.C. 20390	1



David Taylor Naval Ship Research & Development Center Bethesda, Maryland 20034	1
Naval Ocean Systems Center San Diego, California 92132	1
Naval Intelligence Support Center 4301 Suitland Road Washington, D.C. 20390	1
Naval Electronics Systems Command 2511 Jefferson Davis Highway Arlington, Virginia 20360	1
Naval Underwater Systems Center Code SA33 New London, Connecticut 06320	1
Naval Ship Engineering Center Hyattsville, Maryland 20782	1
Naval Coastal Systems Laboratory Panama City, Florida 32401	1
Naval Air Systems Command Code 370 Washington, D.C. 20361	1
Naval Sea Systems Command Code 03424 Washington, D.C. 20362	1
Naval Underwater Systems Center Newport, Rhode Island 02840	1
Naval Ordnance Station Indian Head, Maryland 20640	1
Naval Surface Weapons Center Dahlgren, Virginia 22448	1
Anti-Submarine Warfare Systems Project Office Code ASW-137 Department of the Navy Washington, D.C. 20360	1
Office of Naval Research Code ONR-230 800 North Quincy Street Arlington, Virginia 22217	1

	No. of Copies
Office of Naval Research Code ONR-434 800 North Quincy Street Arlington, VA 22217	1
Daniel H. Wagner, Associates Station Square One Paoli, PA 19301	1
Tetra Tech, Inc. 1911 Fort Meyer Dr. Suite 601 Arlington, VA 22209	1
Systems Planning and Analysis 1600 Wilson Blvd. Suite 700 Arlington, VA 22209	1
ORI, Inc. 1400 Spring St. Silver Spring, MD 20910	1
Naval Postgraduate School Monterey, Ca. 93940	
Attn: R. N. Forrest, Code 55Fo	1
A. R. Washburn, Code 55Ws	10
R. J. Stampfel, Code 55	1
Library, Code 55	1